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SCHWARZ'S FORMULA FOR THE EQUATION $\partial w/\partial \bar{z} + b\bar{w} = 0$

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SCHWARZ'S FORMULA FOR THE EQUATION $\partial w/\partial \bar{z} + b\bar{w} = 0$

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ABSTRACT. This article concerns the derivation of a generalized Schwarz's formula for the equation $\partial w/\partial \bar{z} + b\bar{w} = 0$. Various methods are employed for this purpose, involving the use of differential equations, eliptical equations, analytic functions and complex variables. The stated purpose of the article is to express the nucleus of the generalized Schwarz's formula for the canonical regions, i.e. the half-plane and band, in terms of special functions by using the generalized Cauchy's formula. Such a formula could not be derived for the case of the circle. Instead, the nucleus is expressed as a series.

1. Let us look at the differential equation /531*

$$\partial w / \partial \bar{z} + b\bar{w} = 0, \tag{1}$$

where $\partial/\partial \bar{z} = 1/2(\partial/\partial x + i\partial/\partial y)$, $f(z) = u(x,y) + iv(x,y)$, equivalent to a system of two equations of the elliptical type relative to the real functions $u(x,y)$ and $v(x,y)$. The theory of equations (1) was developed in the articles of I.N. Vekua [1,2]. In the case where $A, B, F \in L_p$, $p > 2$, we will use the method of [1], based on the integral representations of solutions by way of the analytic functions of the variable $z = x + iy$. In the case where A, B, F are analytic with respect to x and y we will use the method of [2], based on the continuation of solutions into the region of the complex variables $z = x + iy$ and $\zeta = x - iy$. Thus, there will be times, for example in the case of the constant coefficients A and B in an unbounded region, when the second method will be more advantageous.

By introducing a new function according to the formula $f(z) = w(z) \exp\left\{-\int A(z, \bar{z})\, d\bar{z}\right\}$ and considering $F = 0$ we obtain from (1)

$$\partial f / \partial \bar{z} + Af + B\bar{f} = F, \tag{2}$$

* Numbers in the margin indicate pagination in the foreign text.

where $b = B \exp\left\{2i \operatorname{Im} \int A(z, \bar{z})\, d\bar{z}\right\}$.

Let us examine the case of the constant b. As follows from [1], page 182, each solution of equation (2) satisfies the metaharmonic equation

$$\partial^2 w / \partial z\, \partial \bar{z} - |b|^2 w = 0, \tag{3}$$

and conversely, if $w_1(z)$ is the general solution of equation (3) then $w(z) = w_1(z) - \frac{1}{\bar{b}} \frac{\partial \overline{w_1}(z)}{\partial z}$ yields the general solution for equation (2).

2. On page 308 of [1] the generalized Schwarz formula for an equation similar to type (1) is shown. The purpose of this article is to express the nucleus of a generalized Schwarz formula for the canonical regions, i.e., the half-plane and band, in terms of special functions by using the generalized Cauchy's formula obtained in [3] for the case of equation (2) with a constant coefficient $b \neq 0$. In the case of the circle, unfortunately, we have not been able to derive an analogous formula, and the nucleus is expressed in the form of a series (see formula (20)).

In the case of the half-plane the problem is to find the solution w(z) of equation (2) which is regular in the upper half-plane $y \geq 0$ and satisfying on the real axis the condition Re w = u(x), where u(x) is a given function, and satisfying the relation

$$w(z) = \exp[2|b|r]\, r^{-1/2} o(1), \quad r = |z|, \tag{4}$$

in the vicinity of the infinitely distant point. We will assume here that u(x) is H-continuously differentiated one time and is a member of $L_2(-\infty,\infty)$.

In accordance with [1], page 239, we will seek the solution of w(z) in the form of Cauchy's integral

$$w(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Omega_1(z, t)\, w(t)\, dt - \Omega_2(z, t)\, \overline{w}(t)\, d\bar{t}, \quad w(t) = u(t) + iv(t), \tag{5}$$ /532

where (see [3]).

$$\Omega_1(z, t) = \pi |b| \frac{\bar{z} - \bar{t}}{|z - t|} H_1^{(1)}(2i|b||z - t|), \tag{6}$$
$$\Omega_2(z, t) = -\pi b i H_0^{(1)}(2i|b||z - t|).$$

If we substitute the values $\Omega_1(z,t)$ and $\Omega_2(z,t)$ into formula (5) we will pass to the boundary for $z \to x + i0$, and then we will use the boundary

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condition which for the unknown function v(t) will yield the integral equation

$$\int_{-\infty}^{\infty}\left\{|b|\frac{x-t}{|x-t|}H_1^{(1)}(2i|b||x-t|)-b\cos\sigma\cdot iH_0^{(1)}(2i|b||x-t|)\right\}v(t)\,dt = \\ = u(x)-\int_{-\infty}^{\infty}|b|\sin\sigma\cdot iH_0^{(1)}(2i|b||x-t|)\,u(t)\,dt, \tag{7}$$

where $b = |b|e^{i\sigma}$.

Now we have learned that $H_1^{(1)}$ $(2i|b|\ |x - t|)$ is real and $H_0^{(1)}$ $(2i|b|\ |x - t|)$ is a purely imaginary number (see [4], page 163). We may assume (see [1], page 228) that the desired function v(x) is H-continuously differentiated at each finite point on the real axis.

By carrying out the obvious transformations over the left half of equation (7) we obtain

$$\int_{-\infty}^{\infty} H_0^{(1)}(2i|b||x-t|)\,\varphi(t)\,dt = u(x), \quad -\infty<x<\infty, \tag{8}$$

where

$$\varphi(t) = -\frac{1}{2i}v'(t) - |b|\cos\sigma\cdot iv(t) + |b|\sin\sigma\cdot iu(t). \tag{9}$$

If $\phi(t)$ has been found, then by solving the linear differential equation of the first order (9), we will obtain:

$$v(t) = \exp[2|b|\cos\sigma\cdot t]\left\{C+\int[-2|b|\sin\sigma\cdot u(t)-2i\varphi t)]\times \\ \times\exp[2|b|\cos\sigma\cdot t]\,dt\right\}, \tag{10}$$

where C is a real constant.

Equation (8) is solved by the Fourier method of integral conversions (see [5], pages 400-401). It is simple to prove that for $b \neq 0$ the nucleus of equation (8) is a member of L_1 $(-\infty,\infty)$ (see [6], page 746, formula 14). Direct computations show that (for all real b)

$$\varphi(x) = \frac{i}{4\pi}\int_{-\infty}^{\infty}\sqrt{\alpha^2+4b^2}\exp[-i\alpha x]\,d\alpha\int_{-\infty}^{\infty}u(t)\exp[i\alpha t]\,dt.$$

By substituting this expression in formula (10) we find:

$$v(x) = C \exp[2bx] +$$
$$+ \frac{1}{i} \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial x} H_0^{(1)}(2i|b||x-t|) + bH_0^{(1)}(2i|b||x-t|) \right] u(t)\,dt. \tag{11}$$

Then by substituting (11) for C = 0 in formula (5) we find /533

$$\begin{gathered} w(z) = \frac{1}{2} \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial z} H_0^{(1)}(2i|b||z-t|) + bH_0^{(1)}(2i|b||z-t|) \right\} u(t)\,dt + \\ + \frac{1}{2} \int_{-\infty}^{\infty} u(t)\,dt \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial z} H_0^{(1)}(2i|b||z-x|) - bH_0^{(1)}(2i|b||z-x)|) \right\} \times \\ \times \left\{ \frac{\partial}{\partial x} H_0^{(1)}(2i|b||x-t|) + bH_0^{(1)}(2i|b||x-t|) \right\} dx. \end{gathered} \tag{12}$$

Let us now consider the function

$$w(z) = \int_{-\infty}^{\infty} \left\{ \frac{\partial}{\partial z} H_0^{(1)}(2i|b||z-t|) + bH_0^{(1)}(2i|b||z-t|) \right\} u(t)\,dt. \tag{13}$$

If u(t) vanishes in the vicinity of the infinitely distant point equation (13) will be asymptotic [4]. Furthermore, a direct check shows that this function satisfies equation (2) (see end of paragraph 1) and the boundary condition Re w = u(x) on the axis y = 0. On the other hand it can be represented by formula (12). Consequently formula (12) conforms with formula (13) if u(t) vanishes in the vicinity of ∞. For arbitrary $u(t) \in L_2(-\infty,\infty)$ formula (13) proves to be the passage to the limit.

Formula (13) is in fact the generalized Schwarz formula for equation (2) in the case of the real constant b for the upper half-plane.

In the case of the complex b the formula has a more awkward form.

3. Let us consider the case of the band $0 \leq \text{Im } z \leq h$.

By finding the solution w(z) of equation (2) which satisfies the conditions

$$\text{Re } w = u_0(x) \text{ for } y = 0, \quad u_0(x) \in L_2(-\infty,\infty),$$
$$\text{Re } w = u_1(x) \text{ for } y = h, \quad u_1(x) \in L_2(-\infty,\infty),$$

in the form of Cauchy's integral and reasoning as before we find the system of integral equations:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} H_0^{(1)}(2i\,|b|\,|x-t|)\,\varphi(t)\,dt+\int_{-\infty}^{\infty} H_0^{(1)}(2i\,|b|\,|x-t-ih|)\,\psi(t)\,dt=f_1(x),\\
&\int_{-\infty}^{\infty} H_0^{(1)}(2i\,|b|\,|x-t|\,\psi(t)\,dt+\int_{-\infty}^{\infty} H_0^{(1)}(2i\,|b|\,|x-t+ih|)\,\varphi(t)\,dt=f_2(x),
\end{aligned}
\tag{14}
$$

where

$$
\varphi(t)=-\frac{1}{2i}v_0'(t)-biv_0(t),\qquad \psi(t)=-\frac{1}{2i}v_1'(t)-biv_1(t),
$$

$$
f_1(x)=u_0(x)-|b|\,h\int_{-\infty}^{\infty} H_1^{(1)}(2i\,|b|\,|x-t-ih|)\frac{u_1(t)\,dt}{|x-t-ih|},
$$

$$
f_2(x)=u_1(x)+|b|\,h\int_{-\infty}^{\infty} H_1^{(1)}(2i\,|b|\,|x-t+ih|)\frac{u_0(t)\,dt}{|x-t+ih|};
$$

$w_0 = u_0 + iv_0$, $w_1 = u_1 + iv_1$ are the values of the desired function for $y = 0$ and $y = h$ respectively.

By solving system (14) by Fourier's method we obtain the formula /534

$$
\begin{aligned}
w(z)=\frac{1}{2}\int_{-\infty}^{\infty}\Big\{&\frac{\partial}{\partial z}H_0^{(1)}(2i\,|b|\,|z-t|)+bH_0^{(1)}(2i\,|b|\,|z-t|)+\\
&+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\Big[\frac{\partial}{\partial z}H_0^{(1)}(2i\,|b|\,|z-t_1|)-bH_0^{(1)}(2i\,|b|\,|z-t_1|)\Big]R_1(t_1,t)\,dt_1-\\
&-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\Big[\frac{\partial}{\partial z}H_0^{(1)}(2i\,|b|\,|z-t,-ih|)-\\
&-bH_0^{(1)}(2i\,|b|\,|z-t,-ih|)\Big]R_4(t_1,t)\,dt_1\Big\}u_0(t)\,dt+\\
+\frac{1}{2}\int_{-\infty}^{\infty}\Big\{&\frac{\partial}{\partial z}H_0^{(1)}(2i\,|b|\,|z-t-ih|)+bH_0^{(1)}(2i\,|b|\,|z-t-ih|)-\\
&-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\Big[\frac{\partial}{\partial z}H_0^{(1)}(2i\,|b|\,|z-t_1|)-bH_0^{(1)}(2i\,|b|\,|z-t_1|)\Big]R_2(t_1,t)\,dt_1+\\
&+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\Big[\frac{\partial}{\partial z}H_0^{(1)}(2i\,|b|\,|z-t_1-ih|)-bH_0^{(1)}(2i\,|b|\,|z-t_1+ih|)\Big]\times\\
&\times R_3(t_1,t)\,dt_1\Big\}u_1(t)\,dt,
\end{aligned}
\tag{15}
$$

where

$$\left.\begin{matrix} R_1(x,t) \\ R_3(x,t) \end{matrix}\right\} = \int_{-\infty}^{\infty} \left\{1 \pm \sqrt{\frac{2h}{\pi}} (\alpha^2 + 4b^2)^{1/4} \exp\left[-h\sqrt{\alpha^2 + 4b^2}\right] \times \right.$$

$$\left. \times K_{1/2}\left(h\sqrt{\alpha^2 + 4b^2}\right)\right\} \frac{(2b - i\alpha) \exp\left[i\alpha(t - x)\right] d\alpha}{\left(1 - \exp\left[-2h\sqrt{\alpha^2 + 4b^2}\right]\right)\sqrt{\alpha^2 + 4b^2}},$$

$$\left.\begin{matrix} R_2(x,t) \\ R_4(x,t) \end{matrix}\right\} = \int_{-\infty}^{\infty} \left\{1 \mp \sqrt{\frac{2h}{\pi}} (\alpha^2 + 4b^2)^{1/4} \exp\left[h\sqrt{\alpha^2 + 4b^2}\right] \times \right.$$

$$\left. \times K_{1/2}\left(h\sqrt{\alpha^2 + 4b^2}\right)\right\} \frac{(2b - i\alpha) \exp\left[-h\sqrt{\alpha^2 + 4b^2}\right] \exp\left[i\alpha(t - x)\right] d\alpha}{\left(1 - \exp\left[-2h\sqrt{\alpha^2 + 4b^2}\right]\right)\sqrt{\alpha^2 + 4b^2}}.$$

The top symbol corresponds to the top function and the bottom symbol corresponds to the bottom function.

The case of the unit circle is similarly considered. The solution however is not obtained in a closed (integral) form, but rather by using several infinite series.

In conclusion I wish to express my gratitude to I.I. Danilyuk for the statement of the problem and his assistance in writing this article.

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